

Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect. *Limit of a Function*¹

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Abstract

In the first part of this article [2], the concepts considered were those of propositional definitions - the most common cases of the logical structure of definitions, which kids encounter as early as at preschool age. In high school students meet concepts with quantifiers in definition.

This paper illustrates the methods of teaching and learning such concepts in terms of using counterexamples and software considering the notion of limit of a function. In general, the types of student's activities in studies of limit do not depend on the specifics of functions (whether they are real or complex, uni- or multivariate, etc.), whereas, the supporting models are dramatically different.

1. Univariate case

1.1. Intuitive understanding

A popular intuitive definition of limit of a function looks as follows:

D1 | The limit of $f(x)$, as x approaches a , equals L if we can make the values of $f(x)$ arbitrarily close to (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a .

Answering the question: “When the limit of $f(x)$, as x approaches a , does not equal to L ?” students construct negation of D1 by naturally changing “can” to “cannot” there.

In terms of teaching, this dichotomy is not very fruitful. The typology of counterexamples becomes clear taking into account the prospect of the subsequent study of another fundamental mathematical concept - continuity, based on the notion of a limit. The discontinuity, as negation to continuity, provides us with a rich and concrete selection of types of counterexamples.

In general, the need for propaedeutic of subsequent contents - one of the teaching rules - often delivers fresh pedagogical ideas, including those relating to examples and counterexamples of currently studied concepts.

¹ It is the second part in the series of papers “Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect”.

The model **M1** uses graph of function $f(x)$, and based on understanding of “closeness” to value a as belonging to some small interval with center at a . Fig.1 shows the model², where $f_1(x)$ is a light blue line $y = 0.2(x^3 - 2x^2)/(x - 2)$.

Model includes two variables a and δ , which allow to express the condition of closeness of x to a in form $|x - a| < \delta$ by means of function $f_2(x)$ defined as “ $y = f_1(x)$, if $|x - a| \leq \delta$ ”³ - the dark green subcurve.

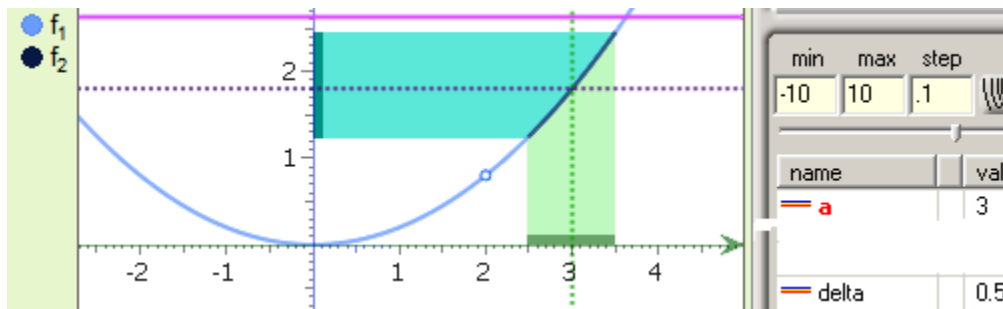


Figure 1

Fortunately, we can use the *VisuMatica*'s possibility to display *domain* and *range* of function (intervals on the coordinate axes). The optional accompany curvilinear trapezoids illustrate their “source”. Model also includes two dotted lines $x = a$, $y = f_1(a)$, and a magenta line $y = b$ - a spare one (until the right time it can be made invisible).

Exploration tasks (Types)

1. Change the value of parameter δ (use the scroll bar). Make it as small as possible. *What happens with graph of function $f_2(x)$? ...with its domain and range? How do you read it from the chart? What is the value of limit at $a = 3$?*
2. Repeat step 1 in case of $a = 2, -1, \dots$
3. *Is the range symmetric with respect to the limit?*
4. *Is the biggest distance of points in range from the limit always less or equal to δ ?*
To check it - select and redefine $f_1(x)$ with a suitable expression.
5. Consider the following cases:
 - a) Select function $f_1(x)$ and redefine it to $y = x + 1/(x - 1)$.
 - Set $a = 0.5$. *What limit has $f_1(x)$ when x approaches 0.5, if any (Fig.2 a,b)?*
 - Set $a = 1$. *What limit has $f_1(x)$ when x approaches 1, if any (Fig.2 c,d)?**What is the principal difference between these cases of $a = 0.5$ and $a = 1$? How one can recognize this difference just by the only graph of initial function $f_1(x)$? Offer five more values of a , where $f_1(x)$ has a limit. Find these limits without *VisuMatica*. Can you guess other value of a , where the limit does not exist?*
 - b) Select function $f_1(x)$ and redefine it to $y = x/2 + [x]$.

² Starting from here all the models constructed by the author's software *VisuMatica*.

³ The inequality should be strict, but we use a legal unstrict version to eliminate the hollow circled at the ends of graph.

- Set $a = 1.5$. What limit has $f_1(x)$ when x approaches 1.5, if any (Fig.3 a, b)?
- Set $a = 2.0$. What limit has $f_1(x)$ when x approaches 2.0, if any (Fig.3 b)?

Change the model to support exploration of one side limits (redefine $f_2(x)$ properly).

- c) Select function $f_1(x)$ and redefine it to $y=(x^4-x^2)/(x^2-1)$, $a = 1$.
 What now is similar and what is different from the previous cases?
 Why the horizontal dotted line disappeared?
 Explain the meaning of the punctured circle and the two white gaps in Fig.4, a).
 “Play” with parameter δ . Has function $f_1(x)$ a limit at 1.0?

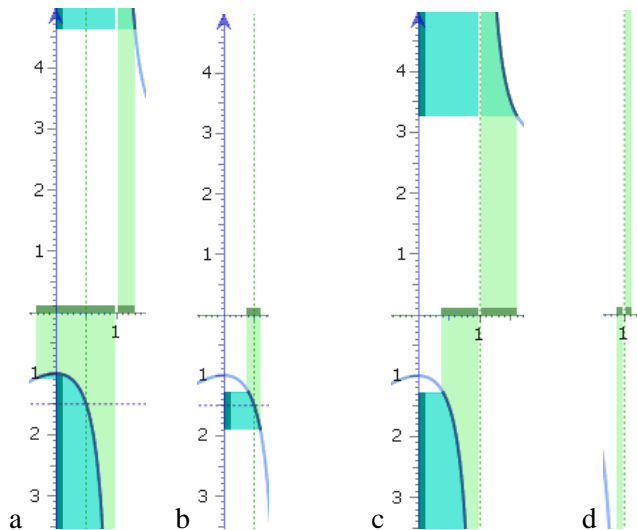


Figure 2. a) $\delta = 0.8$, b) $\delta = 0.1$, c) $\delta = 0.6$, d) $\delta = 0.1$

If your answer is “Yes” then

What is the limit’s value? Is the limit equal to $f_1(a)$?

Change the value of parameter a to locate the magenta line $y = b$ in the expected position of the disappeared horizontal line $y = f_1(a)$ (Fig.4 b).

else

Read the definition D1 once more. Pay attention to its ending condition. Have you changed your mind? What is the limit’s value?

- d) Select function $f_1(x)$ and redefine it to $y = 2$ if $x = 1$ else $(x^4 - x^2)/(x^2 - 1)$, $a = 1$.

What now is similar and what is different from the previous task (c)?

Why does the horizontal dotted line appear again??

“Play” with parameter δ . Has function $f_1(x)$ a limit at 1.0?

What is the limit’s value? Is the limit equal to $f_1(a)$?

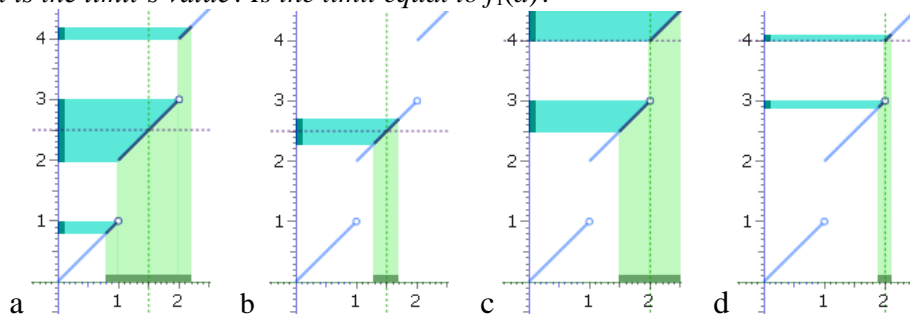


Figure 3. a) $\delta = 0.7$, b) $\delta = 0.2$, c) $\delta = 0.5$, d) $\delta = 0.1$

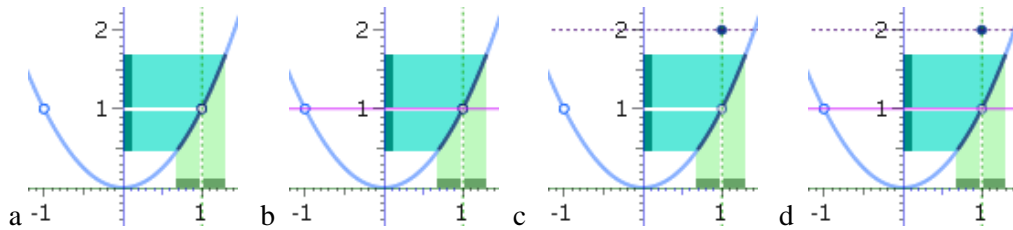


Figure 4

1.2 Definition by Cauchy

The understanding of “closeness” of x to a , expressed by inequality $|x - a| < \delta$ with small enough δ , was sufficient in modeling of intuitive definition of the concept of limit. But, actually, we were looking for two “closenesses”, related to the function $f(x)$: closeness of argument x to a , and closeness of the correspondent $f(x)$ to the value of limit L .

This argumentation in class finalizes by introduction of the “ $\epsilon - \delta$ ” definition:

D2 | The number L is called the *limit* of function $f(x)$ as $x \rightarrow a$ if and only if, for every $\epsilon > 0$ there exists such $\delta > 0$ that $|f(x) - L| < \epsilon$, whenever $0 < |x - a| < \delta$ and $x \in Dom(f)$.

Such logically complicated constructions students meet for the first time. Clarification of definition by means of the following formal logical notation, which includes both universal and existence quantifiers and implication, stresses students even more.

D3 | The number L is called the *limit* of function $f(x)$ as $x \rightarrow a$ if and only if
 $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in Dom(f))(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)$

In case of counterexample, this statement fails.

It can be useful to explain the general role of negation of expressions with quantifiers:

- a) Change the quantifiers (\forall to \exists and \exists to \forall).
- b) Negate the predicate expressions.

Thus, $\lim_{x \rightarrow a} f(x) \neq L \Leftrightarrow (\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in Dom)(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$

In human:

The number L is *not the limit* of function $f(x)$ as $x \rightarrow a$ if exists such $\epsilon > 0$ that for each $\delta > 0$, $|f(x) - L| \geq \epsilon$, whenever $0 < |x - a| < \delta$.

Analysis of “ $\epsilon - \delta$ ” definition D3 with students brings up the following “algorithm” of limit recognition:

1. Select a and an expected limit L . Choose some $\epsilon > 0$.
2. Find such $\delta > 0$, that for all $0 < |x - a| < \delta$ the condition $|f(x) - L| < \epsilon$ remains correct.
3. If a proper δ in step 2 was found
 choose some smaller value of ϵ , say, its half and go to step 2⁴,
 else
 L is not the limit of function $f(x)$ as $x \rightarrow a$.

Fig.5 shows model **M2** for studies of “ $\epsilon - \delta$ ” definition. It includes:

⁴ Pay students’ attention to the infinite amount of such returns if L is really the limit.

1. Variables a , ϵ , δ , and L (at the beginning defined as $L = f_1(a)$).
2. Graph of function $y=x^3/50$ in the role of $f(x)$ - light blue curve f_1 .
3. Epsilon neighborhood related objects: f_2 , defined as $y = f_1(x)$, if $|f_1(x) - L| < \epsilon$ - dark magenta subcurve with its *domain*-light blue and *range*-light green, and light yellow bar $|f(x) - L| < \epsilon$.
4. Delta neighborhood related objects – as in model M1 (Fig.1). Here f_3 is defined as $y = f_1(x)$, if $|x - a| \leq \delta$.

Student interact with model by changing values of its four parameters. As a starting point, model presents the case of L , defined by expression $f_1(a)$ (Fig.5). So, we have only three independent parameters a , ϵ , and δ ⁵. Such “limitation” of $L = f_1(a)$ is very helpful at the beginning of student’s activities: it eases explorations, and it provides an important propaedeutics of the concept of continuity.

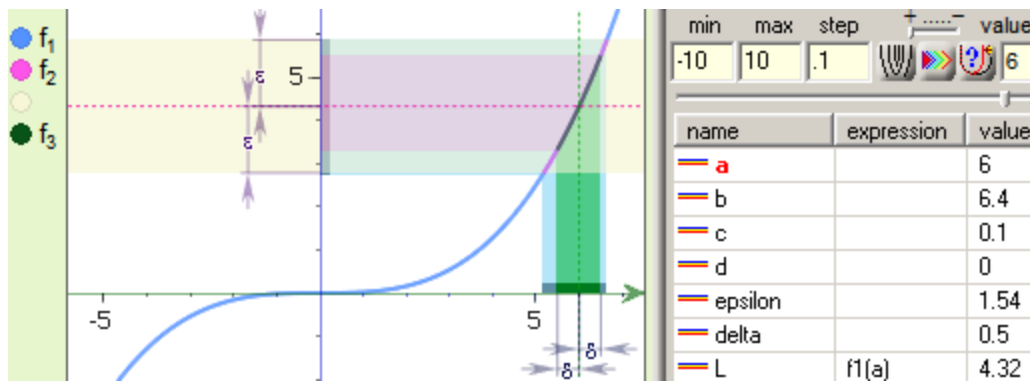
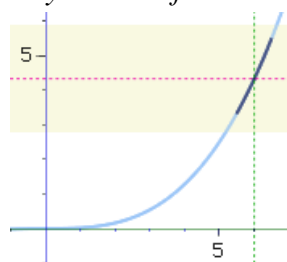


Figure 5

The work with this model on examples and counterexamples of the limit is based on the formulated algorithm and actively refers to model’s elements as analogs of elements of the definition. In result of routine of these activities it becomes clear that we can *facilitate* our model. It will be sufficient to leave it only with the yellow ϵ -bar, graphs of the initial function f_1 and the current f_3 , defined on the δ -interval. We do not need any region (domains, ranges and curvilinear trapezoids). Our new model **M3** (Fig.6) becomes easier to manipulate and explore without loose of its educational potential⁶. Of course, this conclusion and transition to the “light” model M3 is **justly only after detailed exploration with students of the ϵ - δ definition of limit by means of model M2.**



⁵In these terms, L is not a limit if has been found such value of ϵ , that for any arbitrarily small δ we see magenta outside the yellow bar.

⁶In these terms, L is not a limit if has been found such value of ϵ , that for any arbitrarily small δ we see parts of dark green subcurve outside the yellow bar.

Figure 6

After solving examples and counterexamples of mentioned *types* with these models, let us construct a *provocative counterexample*.

We start with a very simple function $y = (x - 2)^2 - 1$ and step-by step transform it up to $y = \sqrt{|(x - 2)^2 + 1|} - 1$ (Fig.7)⁷.

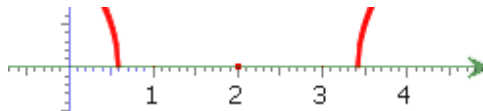


Figure 7

The received graph includes a red isolated point (2, 0). Is $\lim_{x \rightarrow 2} y = \sqrt{|(x - 2)^2 + 1|} - 1 = 0$?

Really, for each $\varepsilon > 0$ and any $0 < \delta < \sqrt{2}$ we have no x in domain of $f(x)$. The δ -neighborhood is empty-there is nothing to check on satisfaction of inequality $|f(x) - L| < \varepsilon$.

To take into account similar situations it is common to include requirements to the domain in definition of the concept of limit of a function. The simplest way, is to require belonging of the whole punctured opened δ -interval to the function's domain $\text{Dom}(f)$. However, *arbitrarily closeness* has a better presentation by means of the notion of accumulation point:

D4 | Point a is called an *accumulation point* of a set S if for every $\varepsilon > 0$ there exists a point $x \in S$ such that $0 < |x - a| < \varepsilon$.

If a is an accumulation point of the domain, then there are infinitely many other neighboring points in domain. Now it is time to update the D2 definition as follows:

D5 | Let a be an accumulation point of the domain of the function f . Then the number L is called the *limit* of function $f(x)$ as $x \rightarrow a$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ so that, whenever $x \in \text{Dom}(f)$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

This definition clarifies the judgement about the previous limit as a counterexample, and the following *pathological* function's behavior as a correct example of a limit.

Consider $\lim_{x \rightarrow 0} \sqrt{x \sin \frac{1}{x}}$. Fig.8 shows graph of function $f(x) = \sqrt{x \sin \frac{1}{x}}$ with different levels of zooming. Function's domain does not include 0. However, the graph condenses as we are approaching 0⁸. It looks like from some zooming moment the whole "area" around the origin belongs to graph.

To catch up craftiness of graph pay students' attention to the radicand. When we graph it separately it becomes clear that radicand has negative values arbitrarily close to zero. Arguments of these values are not in the domain. So, there is no neighborhood of 0, which fully belongs to $\text{Dom}(f)$! Meanwhile, that radicand has positive values arbitrarily close to zero. Arguments of these values are in the domain of $f(x)$. Each neighborhood of 0 includes points that belong to $\text{Dom}(f)$. Thus, zero is an accumulation

⁷ This and the following examples illustrate the modeling strength of *VisuMatica* – an important feature of educational software.

⁸ *VisuMatica* marks it by a punctured origin. *Why?* – The students' answer follows soon.

point of the $\text{Dom}(f)$ and consideration of the limit of $f(x)$ becomes legitimate. Graph of radicand also helps to grasp the value of limit, especially after addition of radicand envelopes $y = \pm x$ and referring to the “sandwich” theorem.

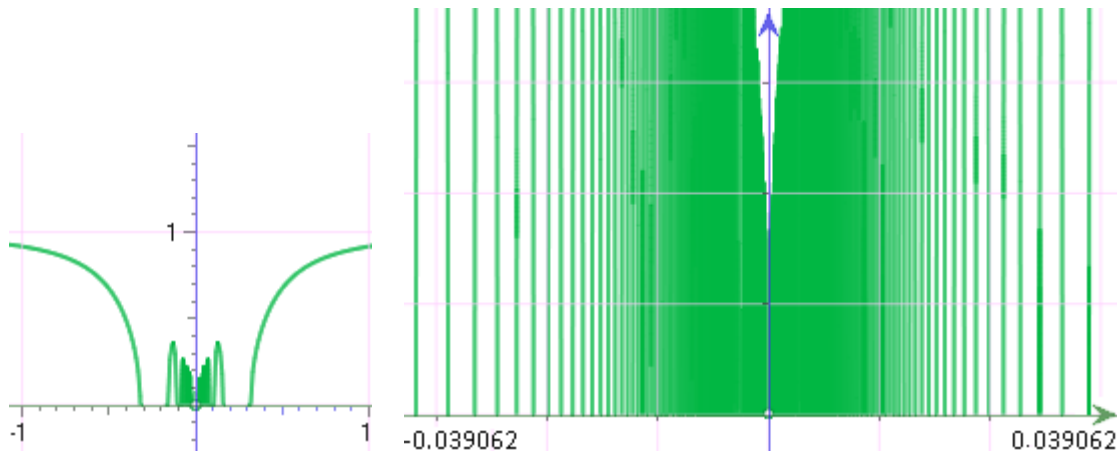


Figure 8

Summarizing together with students their observations of different cases we come to the following scheme (Fig.9). Cases a) - d) present examples of limit, while cases e) - g) present counterexamples.

- *What unites and what differs these cases? If limit is “closeness” then closeness of what is common, and closeness of what distinguishes a)-d) and e)-g)?*

The consideration has no relation to the value of L and ε -interval around it !

We’ve “discovered” the *Cauchy Criterion*:

$$\text{CC} \left\{ \begin{array}{l} \text{The finite limit } \lim_{x \rightarrow a} f(x) \text{ exists if and only if} \\ (\forall \varepsilon > 0)(\exists \delta > 0)(0 < |x_1 - a| < \delta \wedge 0 < |x_2 - a| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon). \end{array} \right.$$

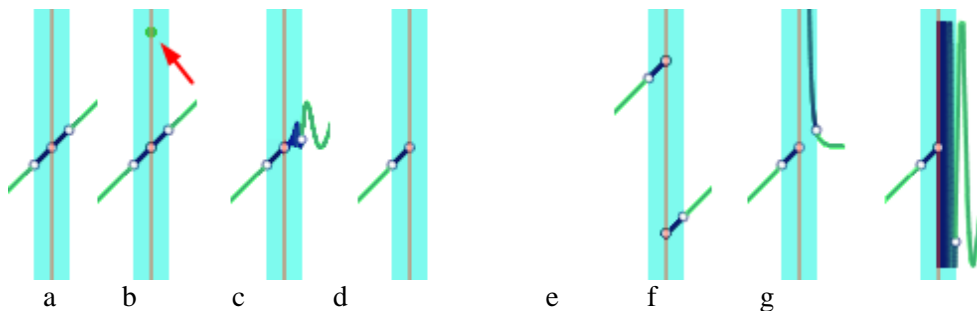


Figure 9

1.3 Definition by Heine

The sequential mechanism in studies of limit finds its definition (by Heine) as follows:

D6 | We call L the limit of function $f(x)$ as $x \rightarrow a$ if for any sequence $\{x_n\}$ converging to a with terms $x_n \neq a$ for all $n \in \mathbf{N}$, the sequence $\{f(x_n)\}$ converges to L as $n \rightarrow \infty$.

So, the software has to provide an ability to generate various sequences $\{x_n\}$ converging to a , and to verify the convergence to L of the corresponding sequences $\{f(x_n)\}$.

The concept of converging sequences seems not so convenient for visualization. Nevertheless, *VisuMatica* models them quite successfully.

M4. Fig.10 shows the model in case of $f_1(x)$, defined as $y = \sin(x)$, and $f_2(x)$, defined as $s(n) = \text{RandomlyTo}(f_1(x), a, 0.05)$, $a = \pi/2$.

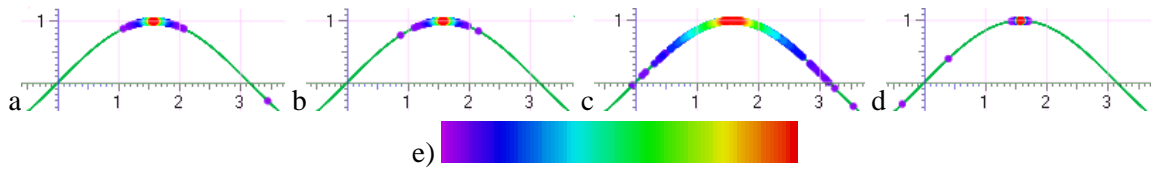


Figure 10

The second expression needs explanation. Its left side $s(n)$ means that it is a definition of a sequence. The right side includes call to a function “*RandomlyTo*”, which initiates generation and drawing of a **random sequence of colored points** $\mathbf{P}_n(x_n, f(x_n))$. Function “*RandomlyTo*” has a three following arguments:

1. Expression of function $f(x)$. In this model it is the initial function $f_1(x)$, i.e $\sin x$.
2. Expression that defines a , to which $\{x_n\}$ converges.
3. Convergency “*speed*” - an optional argument. When avoided its value considered as 0.05. Fig.10 shows sequence for *speed* = 0.05 (a) and (b), lazy convergence for *speed* = 0.3 (c), and greedy one for *speed* = 0.01 (d).

Thus, the abscissas of generated random sequence of points, located on graph of $f(x)$, converge to the calculated value of the second argument.

Color distribution codes the order of points in the sequence. The palette in Fig.10 e) presents this arrangement. Reading colors of the spectrum from left to right we interpret the order of sequence elements as follows: purple and dark blue points present the first elements, the yellow and especially red ones – elements with the most advanced indices.

Each following point of the sequence covers the colors of the previous points located below it.

Repeatedly pressing button F5 refreshes scene and this way generates one more random sequence with the same parameters (compare images a) and b) in Fig.10).

As always, an important question is: *What does it mean $\lim_{x \rightarrow a} f(x) \neq L$?*

First, we rewrite the definition D6 in a formal logical notation:

$$\mathbf{D7} \left| \begin{array}{l} \text{The number } L \text{ is called the } \textit{limit} \text{ of function } f(x) \text{ as } x \rightarrow a \text{ if and only if} \\ (\forall \{x_n \mid x_n \neq a, n \in \mathbf{N}\})(\{x_n\} \rightarrow a \Rightarrow \{f(x_n)\} \rightarrow L) \end{array} \right.$$

Using the role of negation of expressions with quantifiers we get its negation:

$$\lim_{x \rightarrow a} f(x) \neq L \Leftrightarrow (\exists \{x_n \mid x_n \neq a, n \in \mathbf{N}\})(\{x_n\} \rightarrow a \wedge \neg \{f(x_n)\} \rightarrow L)$$

Students can gain experience in studies using some fresh tasks of previous types.

The following two questions sum up these activities.

- *What feature of the repeatedly redrawed scene can be interpreted as existence of the limit and how to read the limit’s value?*

- How to ease this decision and to make the show clearer?

One more *type of students activities* consists in recognition and explanation of images, created by means of current model:

Exploration tasks (Types)

- Observe and explain images in Fig.11 obtained with model M4:

a) $y = \begin{cases} 3, & x = \pi/2 \\ 2\sin x \end{cases}, a = \frac{\pi}{2},$ b) $y = [x], a = 2,$ c) $y = [x], a = 2.04,$
 d), e) $y = \frac{1}{100x^2},$ f), g) $y = \sin 1/x, a = 0$ with different scales.

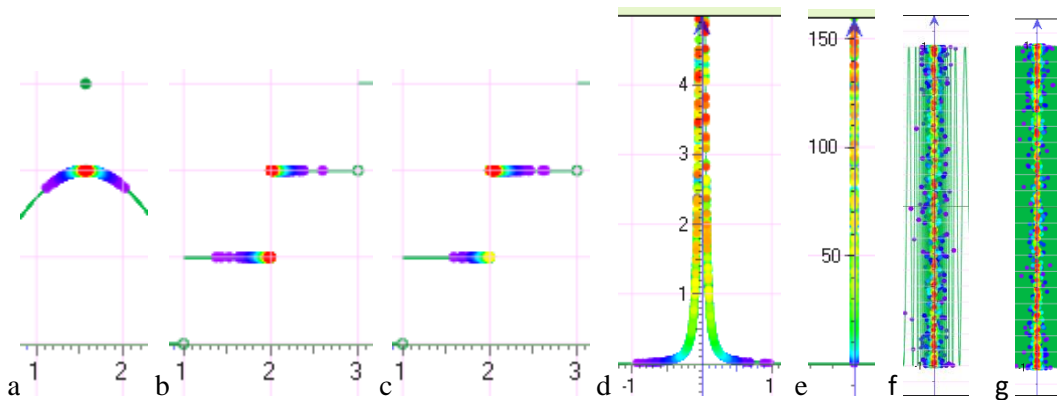


Figure 11

- Compare the two images d) and e) (f) and g)). What did the experimenter, to make the second image more “explanatory”? What is the value of $\lim_{x \rightarrow 0} f(x)$ if any?

Cases f) and g) present a “strange” behavior of the random sequence in our model. *Scaling and repeated redrawing of random distribution (reconstruction of the sequence) does not change the principle - the y-coordinate of red dots remain arranged randomly in the segment [-1, 1].*

Note that the definition by Heine has a constructive nature. In fact, the value of L does not matter: by finding that for all sequences $\{x_n\}$ converging to a - the sequences $\{f(x_n)\}$ **converge to the same value** P , we simply conclude that P is the limit of $f(x)$.

This fact converts the definition by Heine to *Sequential Criterion for Functional Limits*:

HC $\left\{ \begin{array}{l} \text{Given a function } f(x) \text{ and an accumulation point } a \text{ of } \text{Dom}(f), \text{ the following two statements are} \\ \text{equivalent:} \\ \text{a) } \lim_{x \rightarrow a} f(x) = L, \\ \text{b) for any sequence } \{x_n\} \text{ converging to } a \text{ with terms } x_n \neq a \text{ for all } n \in \mathbf{N}, \text{ the sequence } \{f(x_n)\} \\ \text{converges to } L \text{ as } n \rightarrow \infty. \end{array} \right.$

The following *Divergence Criterion for functional limits* is just its corollary:

DC $\lim_{x \rightarrow a} f(x)$ does not exist if and only if one of the following occur:

a) there exist two converging to a sequences $\{x'_n\}$ and $\{x''_n\}$ in $\text{Dom}(f)$ that $x'_n \neq a$ and $x''_n \neq a$ for all $n \in \mathbf{N}$ but $\lim_{n \rightarrow \infty} f(x'_n) \neq \lim_{n \rightarrow \infty} f(x''_n)$.

b) there exists a converging to a sequence $\{x_n\}$ in $\text{Dom}(f)$ that $x_n \neq a$ for all $n \in \mathbf{N}$ for which $\lim_{n \rightarrow \infty} f(x_n)$ does not exist.

Our suspicions about the behavior of function $y = \sin 1/x$, when x approaches to $a = 0$ can not be easily verified by the a) version of *Divergence Criterion* with model M4. Its mechanism for creation of a single random sequence does not fit simultaneous construction of several different sequences. Fortunately, *VisuMatica* allows to highlight extremums. They are looking expressive in the context of our exploration (Fig.12 a,b)). Colored points accent the extremums: points of maximum are shown in blue, while points of minimum – in orange.

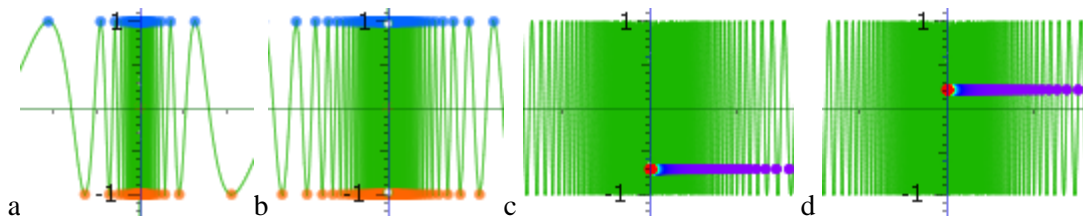


Figure 12

Images a) and b) in Fig.12 present the exploration by changing the x -axis window's boundaries. It becomes clear that the two sets of extremum points include sequences with two different limits -1 and 1 at $a = 0$: the blue points converge to $(0, 1)$, and the orange ones are approaching to $(0, -1)$. On the basis of DC(a) function diverges at 0.

- Find expression that define sequence marked in blue (orange).
Pay students attention to the presence of other sequences $\{(x_n, f(x_n))\}$ with $\{f(x_n)\}$ converging to every possible limit in interval $[-1, 1]$. For example, any line $y = b, b \in [-1, 1]$ crosses the graph of the function at an infinite number of points that can be represented as an infinite constant sequence $\{f(x_n) = b\}$, converging to b .

The following model will help students to handle similar tasks, related to the issue of *Convergence-Divergence Criterion*:

M5. Fig.12 c), d) shows the outputs of model that “solves” the last task and consists of:

- function $f_1(x)$, defined as $y = \sin(1/x)$,
- sequence $s_1(n)$, defined as $x(n) = 1/(2n\pi - \pi/2 + \mathbf{b})$ (invisible), with $\mathbf{b} \in [0, 2\pi]$,
- sequence $s_2(n)$, defined as $y(n) = f_1(s_1(n))$. The sequence is multicolored in the same manner as in M4 (see Fig.10).

- Explore the model - play with the \mathbf{b} parameter and zoom in if necessary.
- Redefine s_1 to approach to the accumulation point from the other side.

In general, function $f(x)$ diverging at some accumulation point a can allow converging sequences $f(x_n)$ with finite $\lim_{n \rightarrow +\infty} f(x_n)$, while $\lim_{n \rightarrow +\infty} x_n = a$, so called **partial limits**.

Playing with the b parameter students discover the fact that the partial limits (Fig.12 c, d) are filling in the whole interval $[-1,1]$.

1.3 Limit Machine

All the considered models intended to materialize students' activities on studies of the concept of limit. Therefore, although they already modeled "almost finished" definitions, nevertheless, the significant component was the accentuation of students' attention to the need for additional actions that replenish the definition (the selection of a suitable value of δ or the numerous generation of random sequences). At some point, the selection of δ begins to annoy.

We suggest students to redesign the model M3 so that it will select an appropriate, the maximum possible value of variable δ "on its own". As a hint, we advise them to take advantage of the initial model, M2 (Fig.5).

Ultimately, the learning activity here was to find within the green interval I of $Dom(f_2)$ ⁹ such symmetric with respect to a dark green subinterval $(a - \delta, a + \delta)$, that the graph of function f_3 , defined on it, will entirely belong to the yellow ε -neighborhood of L .

In *VisuMatica*, function **Dom (index, x_0 , side)** provides access to the information about interval I . Here *index* is an index in the name of function's legend (in our case, the name is f_2 , so *index* = 2), x_0 is the coordinate of the point belonging to the interval (in our case $x_0 = a$), and *side* is the letter: "**l**" for the left border of the interval and "**r**" for the right.

After a brief discussion, students offer the simplest version of a workable model with an automatic finding of the "ideal" value of δ by the given values of a and ε .

This model (Fig.13) includes the following elements:

- Function $f_1(x)$ defined as $y = (x^4 - 2x^3)/(20x - 40) + 2$ graphed by light-blue curve.
- Variables a , ε , and l (at the beginning defined as $l = f_1(a)$).
- Point $P(a, l)$ with projecting perpendiculars onto coordinate axes.
- Epsilon neighborhood related objects: f_2 , defined as $y = f_1(x)$, if $|f_1(x) - P.y| < \varepsilon$ - set as invisible and light yellow bar $|f(x) - L| < \varepsilon$.
- Delta neighborhood related objects - Variables $DomL = Dom(2, a, l)$, $DomR = Dom(2, a, r)$, $\delta = \min(a - DomL, DomR - a)$, and f_3 defined as $y = f_1(x)$, if $|x - a| \leq \delta$, colored in brown-red.

By varying the values of a and ε students are convinced in correctness of the current model, except the case of $a = 2$.

Fortunately, *VisuMatica* provides function **Lim (expression, variable, x_0)**, which calculates the value of $\lim_{variable \rightarrow x_0} expression$.

Redefinition of L as $Lim(f_1(x), x, a)$ fixes the model. It looks like M3 (Fig.6), but finds the value of δ automatically, and properly changes the graph of function f_3 .

⁹ $a \in I$.

Discussion of this model leads to the need for an additional visual emphasis on the following three features:

1. The symmetry of the domain of function f_3 with respect to a .
2. The fact that the graph of function f_3 (brown-red curve) is entirely located in the ε -neighborhood (yellow strip) with possible exception of the point $(a, f_3(a))$.
3. The width 2δ of the domain of function f_3 is the maximum possible.

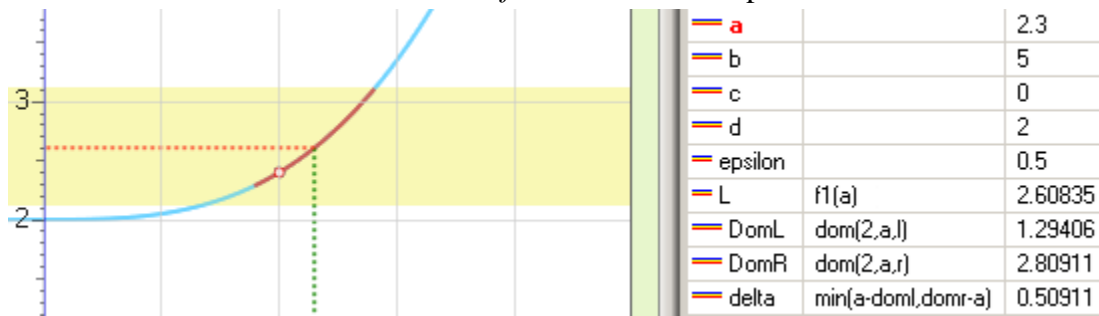


Figure 13

The simplest improvement is the addition of two points $P_1(a-\delta, f_1(a-\delta))$ and $P_2(a+\delta, f_1(a+\delta))$ with their projections onto coordinate axes at the ends of the curve f_3 , especially since the feature of projection is already enabled in the “*OPTIONS*” of the “*Geometry*” dialog (Fig.14, three left images). Vertical lines on both sides are equidistant from the middle vertical (feature 1). The maximum possible distance between them (feature 3) is also confirmed by careful analysis of the picture. Although the horizontal lines of the points P_1 and P_2 are sometimes located on one side of the red line determined by the point $P(a, L)$ nevertheless, all these three lines are always inside the light yellow bar, confirming the fulfillment of feature 2.

Unfortunately, this variant is not perfect: the show fails when one of the abscissas of these points ($a-\delta$ or $a+\delta$) does not belong to the domain of function f_1 . In such case, calculation of the function f_1 fails, resulting disappearance of the point and its projecting lines. Thus, the left point $P_1(2, error)$ becomes invisible in the third image in Fig.14, where $y = (x^2-1)/(x-1)$.

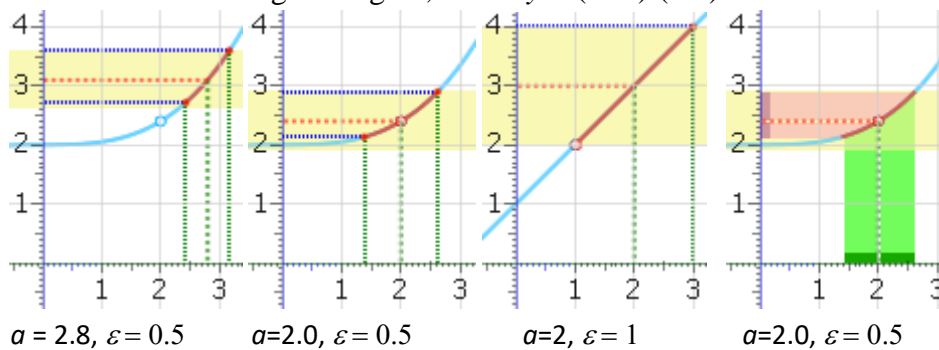


Figure 14

The Domain-Range version of the function f_3 (the right-hand picture of Fig.14) always provides a correct image, but the reddish range part in this case looks redundant. It will be better to emphasize the belonging of all points of the graph of the function f_3 to the yellow bar by outlining it with a rectangle. Finally, attention should be paid to the cases when the function is defined and has a limit at $x = a$, but $\lim_{x \rightarrow a} f_1(x) \neq f_1(a)$. This is easy to do by replacing the vertical segment-perpendicular

to the straight line $x = a$, consisting of points that are excluded from consideration. The final view of the new model **M6** shown in Fig.15.

Fig.16 shows examples of images obtained with its help. Analysis of these and similar images together with the corresponding values of variables is pedagogically essential.

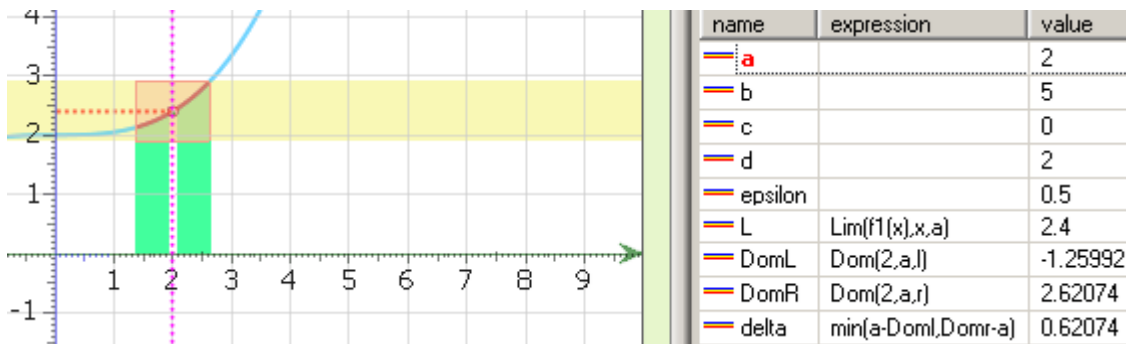


Figure 15

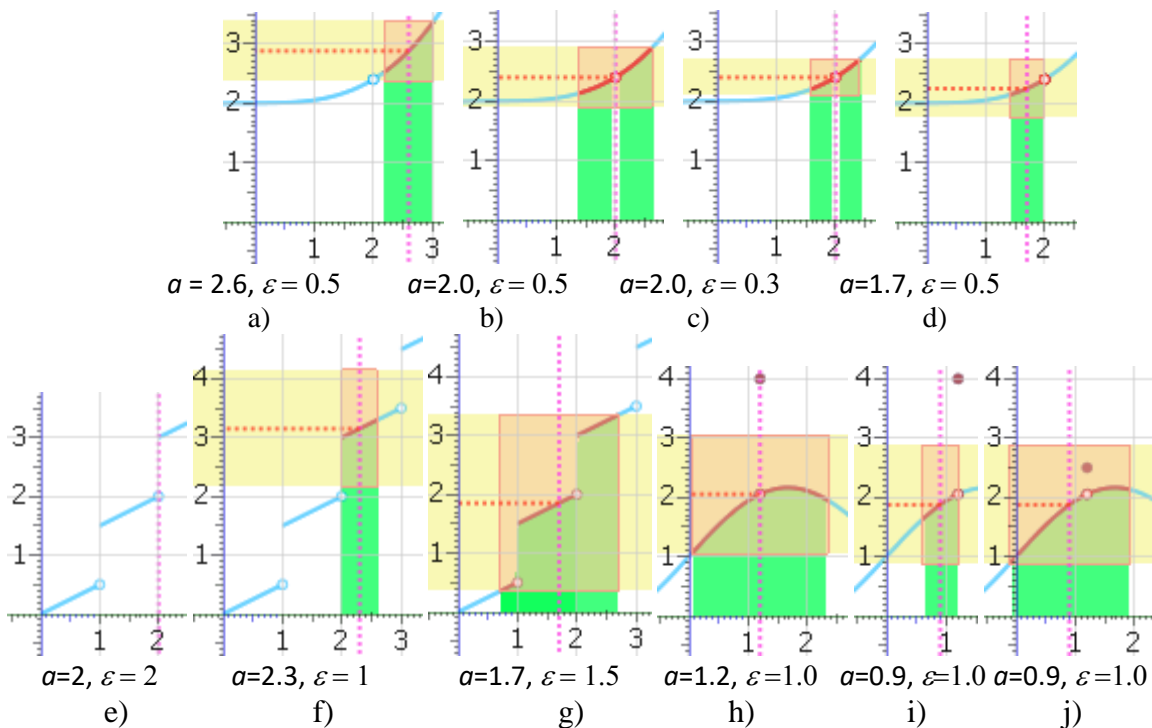


Figure 16. $y = \frac{x^4 - 2x^3}{20x - 40} + 2$ a) - d),

$y = \frac{x}{2} + [x]$ e) - g),

$$y = \begin{cases} \sin x + \frac{x^2}{10} + 1, & x \neq 1.2 \\ 4 \end{cases} \quad \text{h), i),} \quad y = \begin{cases} \sin x + \frac{x^2}{10} + 1, & x \neq 1.2 \\ 2.5 \end{cases} \quad \text{j)}$$

This model will be useful in introducing the concept of continuity of a function.

2. Multivariable case

We need three dimensions to show a surface - graph of a real function of two real variables. To observe the 3D-scene user must rotate it in different ways without guarantee of an adequate

perception of the form of complicated surfaces with singularities. At the same time, the study of the function's limit largely based on examination of counterexamples, which are exactly these, "bad" cases. Let us look for ways to overcome this problem.

2.1 An intuitive definition

D8 | The limit of function $z = f(x, y) = f(P)$ as $P \rightarrow P_0$, equals L if we can make the values of $f(P)$ arbitrarily close to (as close to L as we like) by taking P to be sufficiently close but not equal to P_0 .

The default interface of *VisuMatica* includes all the necessary features to support the working off this definition. After all, we need two things:

- The ability to *differently approach* the point $P(x, y)$ arbitrarily close to the fixed P_0 .
- The ability to *monitor herewith the values of $f(x, y)$* and the way of their changes.

Just enter the function's expression, say $z = x/y$, and move the mouse inside the viewing volume. *VisuMatica* interprets 3D position of the mouse as a point P on the plane xOy and erects a perpendicular to this plane at P up to the graph of $f(x, y)$. The end point is marked by a small ball and a number – the value of $f(P)$. Coordinates of the moving point P displayed continuously in *the status line* on the bottom of *VisuMatica*.

Anyway, this construction serves only the starting, too chaotic phase of exploration.

In order to ease monitoring it is necessary: a) to present P_0 and L , and b) to concretize and make "sensible" both the closeness of the point P to P_0 , and the value of $f(P)$ to L .

Model **M7** includes: variable L , point $P_0(a, b)$ controlled by parameters a and b , expressions " $|f_1(x, y) - L|$ " and " $|mouse, P_0|$ " as based on the mouse position.

Exploration tasks (Types). Use model to:

- Check $\lim_{P \rightarrow (\dots)} \dots = \dots$
- Find $\lim_{P \rightarrow (\dots)} \dots$ or "prove" that the limit does not exist.

Some *inconvenience in moving the mouse manually* becomes apparent. Very often, it "runs away" too far from P_0 . To fix the problem we *add a disc centered at P_0 with radius, controlled by parameter c* . So, one can move the mouse pointer exclusively not far than c from P_0 simultaneously watching the *essential* values of expressions (Fig.17).

Occasionally, the point P_0 , we are approaching to, and/or the disc can be fully or partially hidden by the surface. Never mind, make the surface invisible – all the elements of our construction, involved in the study, will stay visible and active.

Remains one more problem: we *can not guarantee* that in some points P , which we did not pass over while moving the mouse manually, the situation is similar.

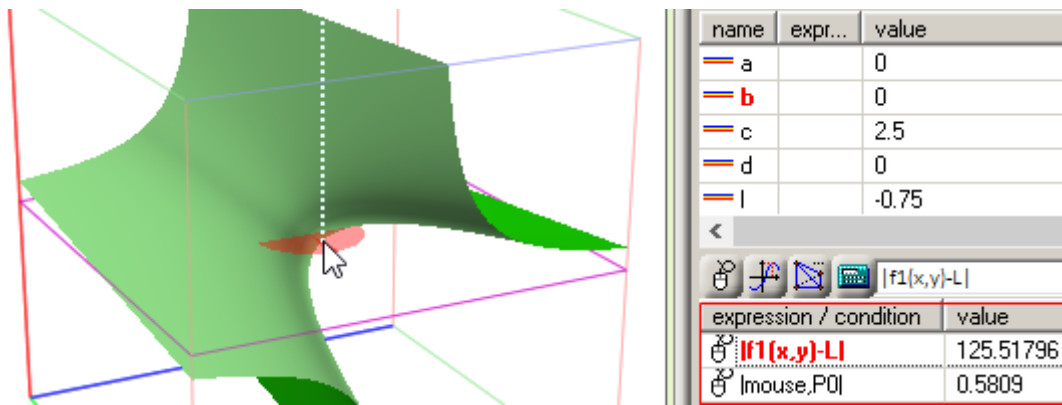


Figure 17

- How to cover all points of the disk, except maybe its center?

The answer comes, while looking with students for possible definitions of the concept of an open punctured disk with center (a, b) and radius r . The classical one we are already using: $\{ (x, y) \in \mathbf{R}^2: 0 < (x-a)^2 + (y-b)^2 < r^2 \}$. Let us improve the model - add $f_2(x, y): z = f_1(x, y), \text{ if } (x-a)^2 + (y-b)^2 < r^2$ and remove the "old" disk from the xOy plane - it only clutters the picture. Unfortunately, the simultaneous display of $f_1(x, y)$ and $f_2(x, y)$ looks problematic because of z -competition: their values coincide in $Dom(f_2)$. In addition, for inspection of surfaces in non-trivial cases, supplementary rotations of space, zooming, etc. are needed, and even these actions are not always sufficient.

Another definition of the disc can be as follows: $\{(x, y) \in \mathbf{R}^2: (x-a)^2 + (y-b)^2 = r^2, r \in]0, c[\}$

To **cover** such region we can use a shrinking circle: $(x-a)^2 + (y-b)^2 = r^2$ with $0 < r < c$ and display the image of this circle, that lies on the surface. Observation of the changing form of this image with reducing radius of the circle, is going to be very helpful.

- To what shape should approach the image of the circle when radius approaches 0?

Model **M8** consists of graph of function $f_1(x)$ (initially $z = x/y$) and a parametric curve $x = r \cos t + a, y = r \sin t + b, z = f_1(r \cos t + a, r \sin t + b), t \in [0, 2\pi]$ -image of a circle.

Fig.18 presents the model in case of $a = -5, b = 3$ and $r = 1$ (left), and $r = 0.3$ (right).

The curve shrinks to a point, whose applicate z presents the limit $\lim_{(x,y) \rightarrow (-5,3)} f_1(x, y)$.

- How the applicates of the points of the image of the circle behave in the case of existence of the limit of the function?

Pay students attention to the full control on parameters of the circle-preimage, to possibility to enforce it to pass through any point by proper selection of the value of radius.

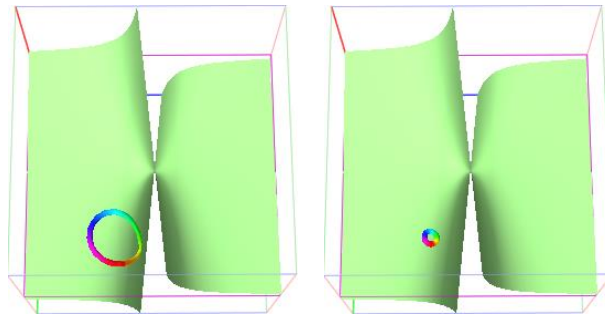


Figure 18

Let us look for a limit of the same function at $P_0(0, 0)$. Set values of both parameters a and b to 0. The result changes drastically. Fig.19 presents the model in case of $r = 2$ (Fig.19 a), and $r = 0.5$ (Fig.19 b). Image of the circle – the multicolored curve – does not shrink to any point while decreasing the value of radius. It seems that points of the curve are spreading along the surface with all possible values of z . This assumption looks reasonable: the initial function includes division by y , whereas ordinate of P_0 equals to 0. The surface has a singularity along $y = 0$. Points $(x,0) \notin \text{Dom}(f)$ and it can seriously affect the surface shape and the limit lengthwise line $y = 0$.

In order to check the guess zoom out the scene (Fig.19 c). The scales' ranges of all axes of the viewing cube are intervals $[-80, 80]$ here. The curve does not spread vertically!

Guilty:... *VisuMatica*. Its defaults resolution of parametric drawing is too low. Change the value in the “*Properties*” dialog, say, to 500, and the show fixes to the assumed one.

The task of looking for a limit at $P_0(40, 0)$ becomes even more complicated. Fig.19 d) displays the model, when $r = 2$ and the resolution of parametric drawing is 50. The curve disappears when r decreases. We have to zoom out to see some parts of curve.

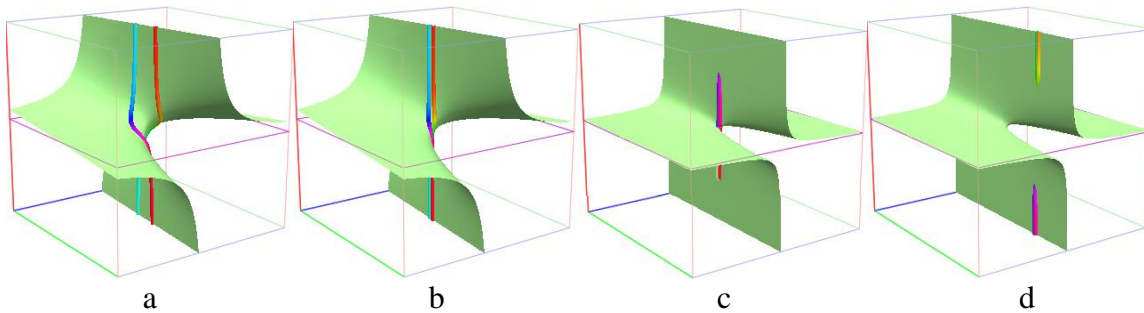


Figure 19

The shape of the curve provides a basis of a useful discussion with class.

One can become curious about this curve. Make graph of $f_1(x,y)$ invisible (Fig.20 a) and rotate the scene around z -axis by pressing left/right buttons. If it still unclear - add a cylinder with radius r , whose rotation axis is parallel to axis z , basis centered at point $(a, b, zMin)$, with $zMin < z < zMax$ (Fig.20 b). Repeat rotations.

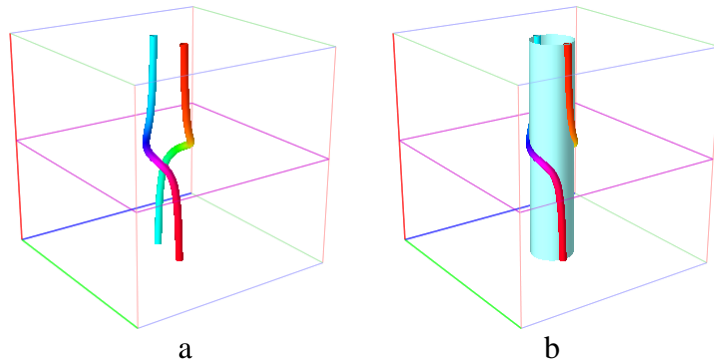


Figure 20

Exploration tasks

- Use current model in the previous exploration tasks. Compare your results with the ones you received with model M7.
- Look for the limits $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{y}$, $\lim_{(x,y) \rightarrow (5,0)} \frac{xy}{y}$, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{y}$, $\lim_{(x,y) \rightarrow (5,0)} \frac{xy^2}{y}$. What is “special” in graphs of these functions? Does it affect existence of the limits?

The curve-image of a shrinking to $P_0(a, b)$ circle in model M8 was rather useful in some cases but not always. Consideration of this curve as spanned on the cylinder, hints idea of a new model. Let us cut this cylinder along its generatrix and deploy it to a plane.

Model **M9** consists of graph of function $f_1(x, y)$ (initially $z = \frac{x}{y - x^3}$) set to be invisible, and scan of a curve on the surface which projection onto xOy plane is a circle centered at (a, b) with radius r . The scan is defined in *VisuMatica* as a function $y = f_1(r \cos t + a, r \sin t + b)$.

Consider $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{y - x^3}$. Fig.21 presents the involute in case of $a = 0, b = 0$, and $r = 1$ (a), $r = 0.00001$ (b). In both cases the x interval is $[0, 2\pi]$ and y interval is $[-10, 10]$.

- What form of graph do we expect when there is a function limit?
- Does the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{y - x^3}$ exist?

Decreasing the value of radius r does not lead to an expected transformation of the graph to a horizontal line. Zooming out along y -axis (it is z in the original 3D scene) just emphasizes the spread of values of the function. The y interval in Fig.21 c) is $[-320, 320]$ while $r = 0.00001$. *The limit does not exist.*

Consider $\lim_{(x,y) \rightarrow (1,0.5)} \frac{x}{y - x^3}$. Fig.21 displays also the involute in case of $a = 1, b = 0.5$, and $r = 1.5$ (d), $r = 0.1$ (e), and $r = 0.00001$ (f). The scale of y axis in all images is $[-5, 5]$. It is clearly seen, that while decreasing radius r the “crazy” curve “calms up” and *becomes extremely close to $y = -2$, to the value of limit we are looking for. (Why?)*

- Does the horizontality of graph guarantee the convergence of $f_1(x, y)$ at (a, b) ?

Use the current model to explore $\lim_{(x,y) \rightarrow (0,0)} \sin \frac{1}{x^2 + y^2}$. Explain.

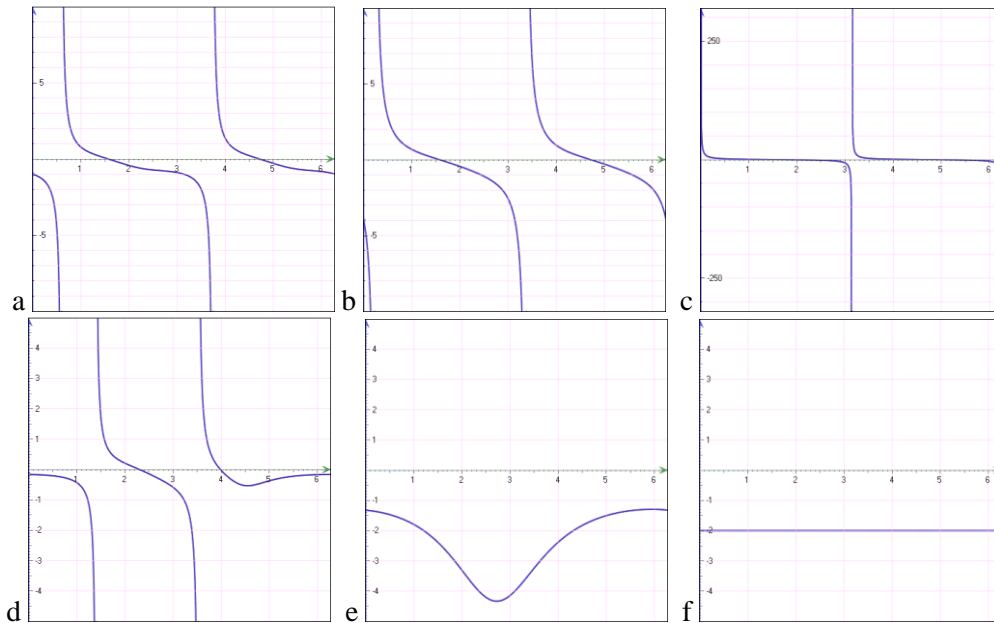


Figure 21

The 3D models we were using before had difficulty in visualization of the value of function as z coordinate (3D perpendicular was the only option to get it). 2D models like the current one are free from this drawback.

To watch the value of $f(P)$ for points P on the circle add expression “f1(radius*cos(c)+a,radius*sin(c)+b)” to the “*expression/condition*” window. Define the c parameter in the boundaries of $[0, 2\pi]$, select the c parameter, and change its value by the scroll bar. It will give an additional, numeric impression in parallel to the graph.

As previously noted, the main disadvantage of naive definition D8 is that practically impossible to follow it: there are infinitely many paths to approach the point P_0 , and, accordingly, an infinite number of ways of behavior of the corresponding values of the function. It is impossible to pass them all.

2.2 Definition by Cauchy

Let us see how the precise definitions can improve our studies.

By analogy with the concept of *neighborhood of point* x_0 as an interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 on the coordinate axis

D9 | We call *neighborhood of point* $P_0(x_0, y_0)$ in the plane Oxy the interior of a disk centered at P . A neighborhood with radius δ is called δ -neighborhood.

Highlight the fact that distance from any point $P(x, y)$ belonging to the δ -neighborhood of the point $P_0(x_0, y_0)$ is less than δ .

D10 Number L is called the *limit* of function $z = f(x, y) = f(P)$ as $P \rightarrow P_0$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(P) - L| < \varepsilon$, whenever P belongs to δ -neighborhood of the point P_0 (except, maybe, the point $P_0(x_0, y_0)$ itself).

$$\text{We write } \lim_{P \rightarrow P_0} f(P) = L \text{ or } \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = L$$

From the definition follows that if L is the limit of $z = f(P)$ then the difference $f(P) - L$ is arbitrarily small when the point P arbitrarily unlimitedly approaches to P_0 .

The model we are going to create should present and allow dynamically acting with two main elements of the definition:

1. The inequality $|f(P) - L| < \varepsilon$.
2. The δ -neighborhood of point P_0 .

Although, this configuration seems not trivial, the model can be rather simple.

Model **M10** (Fig.22) consists of:

- Graph of function $f_1(x)$ (say, $z = x/y$),
- Additional variables: l , *epsilon*, x_0 , y_0 , *delta*,
- Two semitransparent planes $y = L - \textit{epsilon}$, $y = L + \textit{epsilon}$, which bound all the space $|f(P) - L| < \varepsilon$,
- Cylinder, with axis parallel to the axis z , basis centered at point $(x_0, y_0, zMin)$ and height equal to $zMax - zMin$. This cylinder bounds all the 3D points, whose projection onto the plane xOy lie in the δ -neighborhood of point P_0 .

Fig.22 (left) presents checking of $\lim_{(x,y) \rightarrow (0,0)} f_1(x, y)$. Here $x_0 = 0$, $y_0 = 0$, $L = 0$, $\varepsilon = 2$, $\delta = 2$.

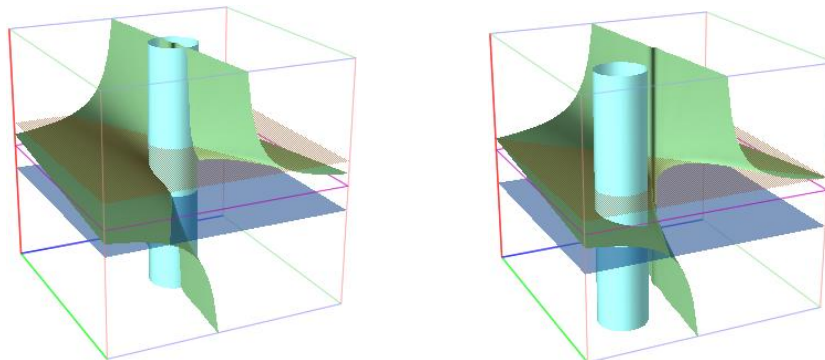


Figure 22

- *How one can use this model to check whether L is the limit of $f_1(x, y)$ at $P_0(x_0, y_0)$? For every $\varepsilon > 0$ should be possible to find such proper value of $\delta > 0$ that all the piece of surface inside cylinder will lay between the two planes.*
- *How to verify this? After all 3D scene can be very complicated and hardly observable. You just have to look into the cylinder from the top and bottom, if the entire surface inside the cylinder resides in the space between the translucent planes, then, when viewed from above, all it will be shaded by reddish tint, and, when viewed from below, - by a bluish, correspondingly to the colors of planes!*

- *Maybe it will be easier to set the planes opaque and make sure that when viewed from above and below our surface is not visible inside the cylinder, only visible the disks of planes?*
The model will not be sufficiently diagnostic: it will miss the difference between the case of “all inside” the ε -layer and the case of counterexample, where “all outside” the viewing volume, but inside the cylinder.

Following the recommendations we immediately discover, that the guess that $\lim_{(x,y) \rightarrow (0,0)} f_1(x,y) = 0$ is wrong: the green pieces of surface inside cylinder are seen already at the top and bottom facets of the viewing volume **does not matter** how much we reduce the value of radius δ . (**Let us recall negation of the limit: ... $\Leftrightarrow (\exists \varepsilon > 0)(\forall \delta > 0) \dots$...**)

Fig.22 (right) presents the model in checking of $\lim_{(x,y) \rightarrow (5,-5)} f_1(x,y) = -1$. Here $x_0 = 5$, $y_0 = -5$, $L = -1$, $\varepsilon = 2$, and $\delta = 2$. -...a promising scene.

Let us look at the structure from top (Fig.23 left) and bottom (Fig.23 right). -Looks good. Even with such large values of ε and δ we see the full tint of surface inside the cylinder.

- *Can we conclude that -1 is really the limit?*

The definition D10 claims: ... $\Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0) \dots$

One can try to reduce the value of ε , and to choose the proper value of δ in order to confirm the belonging of all points of the surface inside cylinder to the ε -layer. Unfortunately, this testing sequence will never end if L is really the limit of the function at P_0 . So, *we are unable to come to a final conclusion.*

Our model allows making a *final conclusion only in the negative case of counterexample*, when L is not a limit, as it was done in the previous experience.

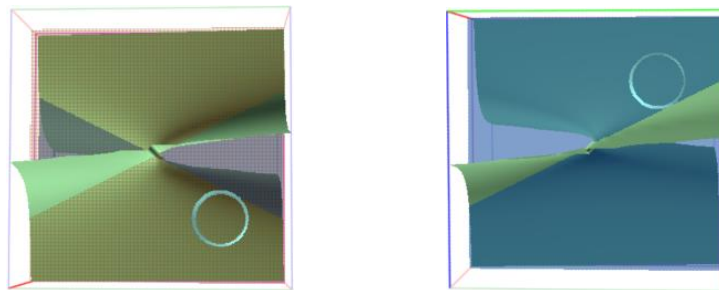


Figure 23

But, what if to try to combine the ε -layer with curve - image of the circle of radius δ from the previous model?

Fig.24 presents different options of such show: with/without graph of $f_1(x,y)$ and multi/single-colored. The first two here show the case of $x_0 = 0$, $y_0 = 0$, $L = 0$ and rest – the case of $x_0 = 5$, $y_0 = -5$, $L = -1$.

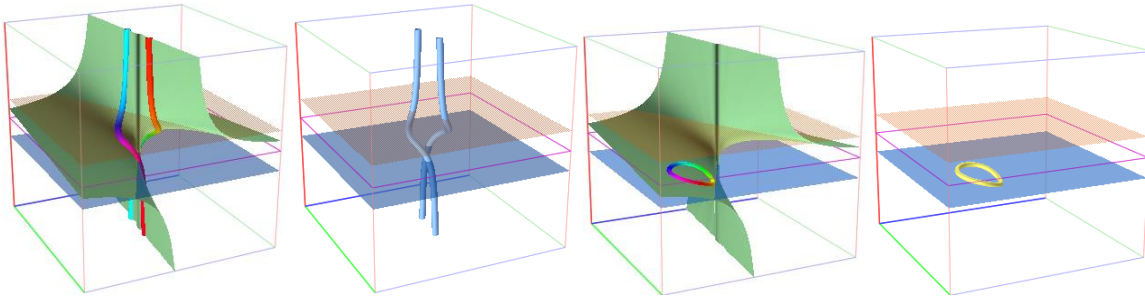


Figure 24

Comparison of this model with the original one, their advantages and disadvantages, can be useful in formation of the concept of limit.

This discussion brings an idea of one more model, which displays only the relevant piece of surface instead of curve.

The following new model **M11** implements this idea. It includes:

- Graph of function $f_1(x, y)$, say, $z = x y$,
- Additional variables: $L, \varepsilon, x_0, y_0, \delta$,
- Two semitransparent planes $y = L \pm \varepsilon$, which bound all the space $|f(P) - L| < \varepsilon$,
- The disk in xOy plane with center at $P_0(x_0, y_0)$ and radius δ ,
- The green image of this disk—the piece of surface defined as “ $f_1(x, y)$, if $(x-x_0)^2 + (y-y_0)^2 < \delta^2$ ”.

Fig.25 shows the case of $x_0 = 1.5, y_0 = 1, L = 1.5, \varepsilon = 1, \delta = 1.9$ (left) $\delta = 0.4$ (right). In accordance to the 3D axes' orientation the piece of surface—the image of the disk has four-color tints:

1. The original color,
2. The color of points inside the ε -layer: reddish green,
3. The color of points under the layer seen through both planes,
4. The color of points under the layer seen through lower plane: bluish green.

All the points in Fig.25 (right) have the reddish green tint. Hence, the whole piece belongs to the ε -layer.

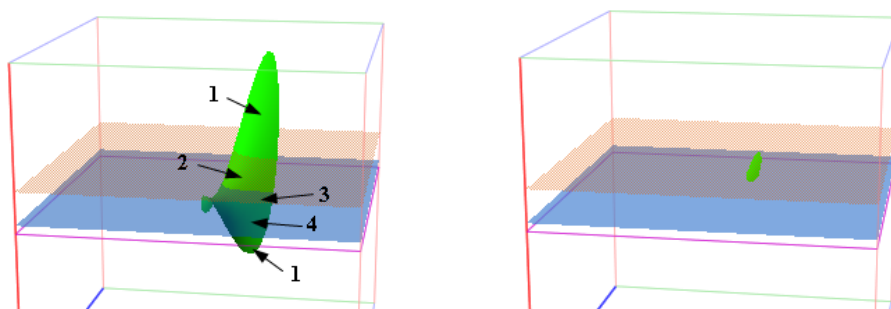


Figure 25

2.3 Definition by Heine

D11 We call L the limit of function $z = f(x, y) = f(P)$ as point $P(x, y) \rightarrow P_0(x_0, y_0)$ if and only if for any sequence $\{P_n(x_n, y_n)\}$ converging to $P_0(x_0, y_0)$ with terms $P_n(x_n, y_n) \neq P_0(x_0, y_0)$ for all $n \in \mathbf{N}$, the sequence $\{f(x_n, y_n)\}$ converges to L as $n \rightarrow \infty$.

So, the software has to provide an ability to generate different sequences $\{P_n(x_n, y_n)\}$ converging to $P_0(x_0, y_0)$ and to check if the correspondent sequence $\{f(x_n, y_n)\}$ converges to L as $n \rightarrow \infty$.

VisuMatica allows generating of $\{(x_n, y_n, f(x_n, y_n))\}$ sequences. They will simulate the different passes similarly to the univariate case.

Model **M12** includes two visible objects:

- Graph of function $z = f_1(x, y)$, say, $z = \sin x + \cos y$.
- Graph of a sequence $s(n) = \text{RandomlyTo}(f_1(x, y), (x_0, y_0), \text{rate})$.

Call to function “*RandomlyTo*” generates a random sequence of 3D points $\{(x_n, y_n, f_1(x_n, y_n))\}$, wherein the sequence $\{(x_n, y_n)\}$ converges to (a, b) as $n \rightarrow \infty$. All these points lie on the surface of graph. The last argument *rate* sets the “laziness” of convergence. The lower the *rate*, the faster $\{(x_n, y_n)\}$ converges to (x_0, y_0) .

The first two images in Fig.26 display the model with $x_0 = -0.8, y_0 = 0$, and *rate* = 0.25. These two images correspond to two different random sequences.

The generation of random sequences with specified values of parameters x_0, y_0 , and *rate* carried out by simple pressing the button F5.

The 3rd image shows the model with *rate* = 0.05, and the 4th, - when *rate* = 0.0001.

Color distribution codes the order of points in the same way as before (Fig.10 e).

Pay students attention to the location of red point(s), while generating different sequences with decreasing value of the “rate”, and the meaning of stability of this location like in Fig.27.

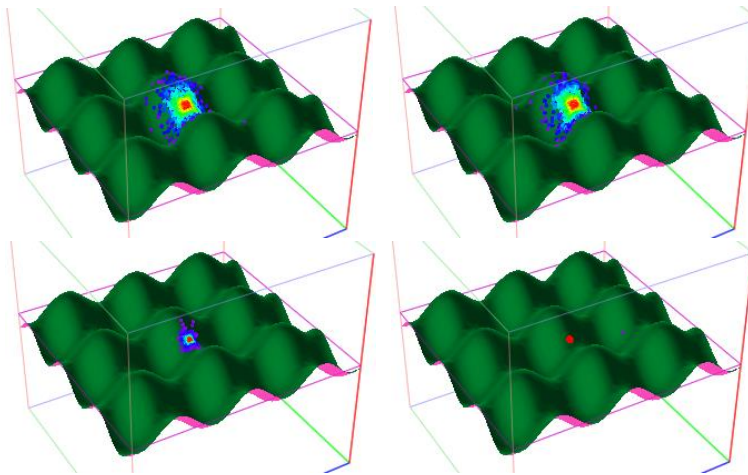


Figure 26

Ease of management allows to quickly and thoroughly probe the behavior of possible sequences, and with a high degree of reliability, to conclude on the convergence of function to the limit.

Consider the already studied example of $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{y}$. Redefine f_1 , set $x_0=0, y_0=0$ (Fig.27).

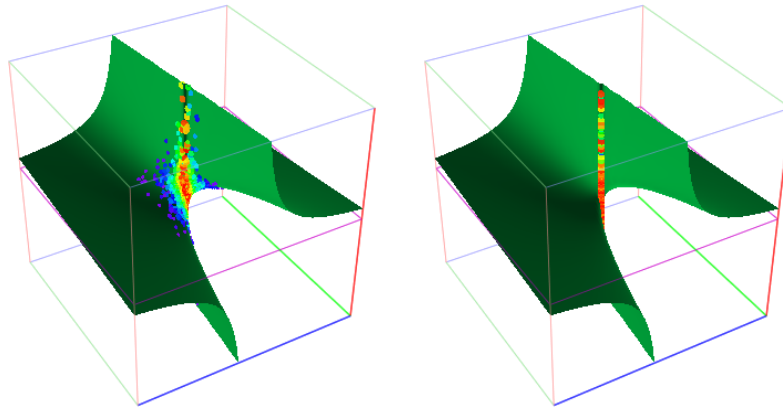


Figure 27. $rate = 0.25$ (left), and $rate = 0.000001$ (right)

“Speeding up” by decreasing the value of $rate$ does not change the situation – red points remain widely distributed over the surface.

With the same suspicions as in the univariate case, we refer to the *Divergence Criterion* (**DC**, section 1.3). It fits the bivariate case, and includes the following statement:

$$\mathbf{DC}^{3d} \left[\begin{array}{l} \lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) \text{ does not exist, if there are two sequences } \{f(x'_n, y'_n)\} \text{ and } \{f(x''_n, y''_n)\}, \\ \text{converging to different limits.} \end{array} \right.$$

Fortunately, function *RandomlyTo* has an option to generate random sequences of points $f(x_n, y_n)$ lying in a vertical plane $y = k(x-x_0)+y_0$, which passes through the point (x_0,y_0) . The additional argument of this function defines the *angle* - $\arctan k$ - in degrees:

RandomlyTo (*functional Expression, point, angle, rate*)

Students add parameter *angle*, redefine $s(n)$ as *RandomlyTo*($f_1(x,y), (x_0,y_0), angle, rate$) and perform the following activities:

1. Set value of *angle* to some number in interval $[0, 360]$.
2. Set rate to 0.25 and repeatedly press F5.
3. Change the *rate* to a lower value and repeat pressing F5.
4. Go to step 1.

Fig.28 displays the model with $rate = 0.25$ and angle: 0° (a), 7° (b), 30° (c), and 90° (d).

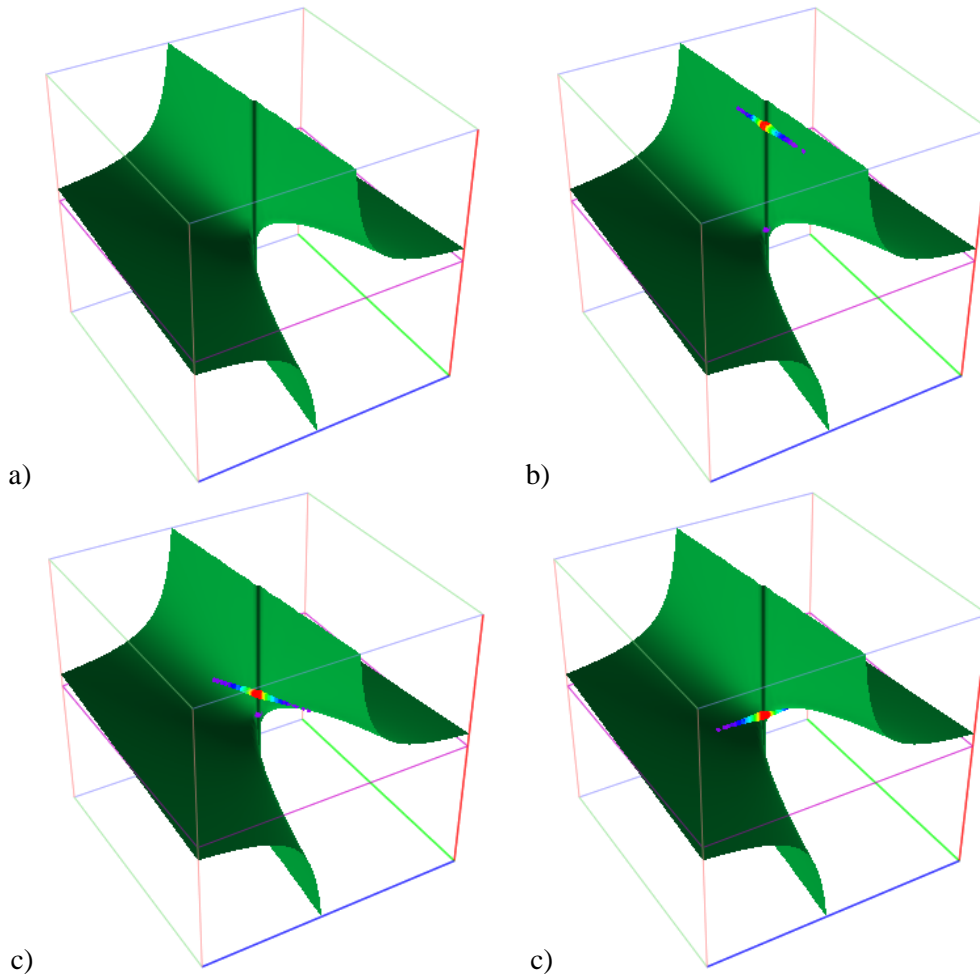


Figure 28

Help students:

- To notice that repeated generating of the random sequences does not matter – points' projection onto xOy plane remain on the same line. (*How to make sure of this?*). The red region is left essentially unchanged.
- To make proper assumptions about the convergence of these sequences.
 - Prove that $\lim_{(x,y) \rightarrow (0,0)} x/y$ does not exist using the experience of exploration.
 - Why Fig.28 a) does not include any point of a random sequence?

Discussion about *the unique arrangement of random sequences on horizontal lines* and its application in solution we are looking for can be interesting and informative.

- Rewriting initial equality $z = x/y$ in form $x = yz$ explains the surface geometric nature: It is a saddle-shaped *hyperbolic paraboloid* with a gap: it does not include points whose $y = 0$ (denominator in the initial expression). From $x = yz$ we immediately conclude that all these points completely fill the z -axis (the dark vertical line in Fig.28 from proper point of view looks as a crack).
- *Hyperbolic paraboloid* is a classical example of a *ruled surface*, through every point of this surface passes a straight line that lies on it.

- Setting z to a definite value, say a , we get equation of the contour line on level a and it is $a = x/y$ or $x = ay$ or $y = 1/a * x$ – clearly a straight rulling line – the *directrix* of our surface. Define this line by two expressions: “*Point(xMin,xMin/a,a)*” and “*Point(xMax,xMax/a,a)*”. Animate parameter a and make sure that this line is the directrix of the surface and at a certain point of time comprises all the points of a random sequence, predetermined by the direction angle. For each horizontal directrix there is a direction angle $atan(1/a)$ that also defines the proper sequence, and *vice versa* - for each sequence there exists a proper directrix, which includes all its points, except $a = 0$. So, we are ready to make a final conclusion: since all the points of any random sequence, defined by the angle of direction, belong to one and the same horizontal line (fixed z), then the limit point should also lie on this line and have the same z -value. In other words, we found an infinite number of sequences, whose projections onto xOy plane $\{(x_n, y_n)\}$ converge to (x_0, y_0) , but the correspondent sequences of function values $\{f_1(x_n, y_n)\}$, being constant and thus converging, have different limits: these constant values of a . Thus, the $\lim_{(x,y) \rightarrow (0,0)} x/y$ does not exist.

Redefine $f_1(x, y)$ back to $z = \sin x + \cos y$ and repeat the exploration.

- What is different in behaviour of the sequences? What this difference means?
- Does it guarantee the existance of $\lim_{(x,y) \rightarrow (0,0)} \sin x + \cos y$?!

In some cases, the *Divergence Criterion for functional limits* works best when considering 3D sequences $\{x_n, y_n, f(x_n, y_n)\}$, whose projections $\{x_n, y_n\}$ on the plane xOy are arranged along certain curves, - and not straight lines at different angles. *VisuMatica* supports construction of such sequences by the following syntax:

RandomlyTo(f(x,y),x0,g(x),rate), RandomlyTo(f(x,y),t0,ParametricCurveName,rate)

Conclusions

In this paper, we show the role and place of counterexamples and effectiveness of software in studies of the concept of limit of a function as an example of a concept with quantifiers in definition. Those presented ideas were realized in the author’s noncommercial software *VisuMatica*, which facilitates the creation of even intricate models by the students themselves¹⁰.

Supplementary Electronic Materials

Videos with animations: <https://sites.google.com/view/counterexamples-in-math-edu-2/>

¹⁰ As a rule, students just load ready-made models using popular software (*it's difficult to create models by themselves; it saves time*). With that is lost the formation of an important human ability to create the necessary means to achieve the desired goal.

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